ON THE RATIO OF THE AREA OF A GIVEN TRIANGLE TO THAT OF AN INSCRIBED TRIANGLE.

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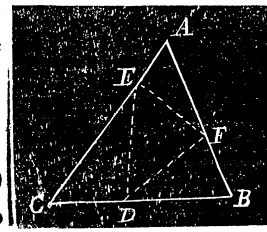
LET us represent the sides BC, AC, AB, respectively by a, b, c; CD by $a\alpha$, AE by $b\beta$, BF by $c\gamma$; the area of the triangle ABC by $\varDelta$ and that of the inscribed triangle DEF by $\varDelta'$.



We easily find triangle $CDE = \alpha(1-\beta).\varDelta$; triangle $AEF = \beta(1-\gamma).\varDelta$; triangle $BDF = \gamma(1-\alpha).\varDelta$. Therefore

$$\frac{\varDelta'}{\varDelta} = 1-\alpha(1-\beta)-\beta(1-\gamma)-\gamma(1-\alpha)$$
$$= 1-(\alpha+\beta+\gamma)+(\alpha\beta+\alpha\gamma+\beta\gamma) \quad \text{(I)}$$

If the three transversals drawn from A, B, C, to the assumed points D, E, F, intersect in one point we have the relation

$\alpha\beta\gamma=(1-\alpha)(1-\beta)(1-\gamma)$, whence $1-(\alpha+\beta+\gamma)+(\alpha\beta+\alpha\gamma+\beta\gamma)=2\alpha\beta\gamma$.

In this case therefore

$$\frac{\varDelta'}{\varDelta} = 2\alpha\beta\gamma. \qquad \text{(II)}$$

1. *Let the three transversals be the medial lines*, then $\alpha = \beta = \gamma = \frac{1}{2}$, and according to (II)

$$\frac{\varDelta'}{\varDelta} = \tfrac{1}{4}.$$

2. *Let the transversals be the bisectors of the angles.* From $CD : BD = b : c$, we get

$$CD = \frac{ab}{b+c}, \therefore \alpha = \frac{b}{b+c}, \text{ and similarly } \beta = \frac{c}{a+c}, \gamma = \frac{a}{a+b};$$

$$\therefore \frac{\varDelta'}{\varDelta} = \frac{2abc}{(a+b)(a+c)(b+c)}, \text{ by (II)}.$$

3. *Let the transversals be the altitudes*

$$CD = b\cos C,\ AE = c\cos A,\ BF = a\cos B.$$

$$\therefore \alpha = \frac{b}{a}\cos C,\ \beta = \frac{c}{b}\cos A,\ \gamma = \frac{a}{c}\cos B;$$

$$\therefore \frac{\varDelta'}{\varDelta} = 2\cos A\cos B\cos C = \frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b+c^2-a^2)}{4a^2b^2c^2}.$$

4. *Let the transversals intersect each other in one point and make equal angles with the radii.* (Vide Prob. 245, No. 1, Vol. VI.)

$$CD = \frac{ab^2}{b^2 + c^2},\ AE = \frac{bc^2}{a^2 + c^2},\ BF = \frac{a^2c}{a^2 + b^2};$$

$$\therefore\ \alpha = \frac{b^2}{b^2 + c^2},\ \beta = \frac{c^2}{a^2 + c^2},\ \gamma = \frac{a^2}{a^2 + b^2};$$

$$\therefore\ \frac{\varDelta'}{\varDelta} = \frac{2a^2b^2c^2}{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)}.$$

Compare this result with that in 2.

5. *Let the points D, E, F, be the feet of the perpendiculars let fall from the centre of the inscribed circle.*

Denote the radius of the inscribed circle by ρ and put $\frac{1}{2}(a+b+c) = s$; then $\alpha = (\rho \div a) \cot \frac{1}{2}C$, $\beta = (\rho \div b) \cot \frac{1}{2}A$, $\gamma = (\rho \div c) \cot \frac{1}{2}B$. Therefore

$$\frac{\varDelta'}{\varDelta} = \frac{2\rho^3}{abc} \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C = \frac{\rho^2 s}{abc} = \frac{2(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{(a+b-c)(a+c-b)(b+c-a)}{4abc}.$$

Compare this result with that in 3.

FIVE GEOMETRICAL PROPOSITIONS.

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I. LET A, B, C, D, &c., be the angular points of a regular polygon of n sides, and let AB, one of the equal sides, aqual unity; then will AB be contained once in AC, the chord which contains *two* of the equal sides, with a remainder which call x. Then is $\sqrt{(1-x)}$ = one side of a polygon of $2n$ sides inscribed in a circle whose radius is *one*.

II. AB will be contained twice in AD, the chord which contains *three* of the equal sides, with a remainder which call y. Then is $\sqrt{(1-y)}$ = one side of a polygon of n sides inscribed in a circle whose radius is *one*.

III. If the polygon be a Nonagon, AB will be contained twice in AE, the chord which contains *four* of the equal sides, with a remainder which call x. Then is $\sqrt{(1-x)}$ = one side of a polygon of 18 sides inscribed in a circle whose radius is *one*.

IV. If in Prop. II the polygon be also a Nonagon, then is

$$\sqrt{(1-x)} = x - y.$$

V. If the polygon be a Decagon, AB will be contained twice in AD, the chord which contains three of the equal sides, with a remainder which call z. Then is $\sqrt{(1-z)} = z$ = one side of a decagon inscribed in a circle whose radius is *one*.